POLYNOMIAL SOLUTIONS OF THE EQUATIONS OF MOTION OF A BODY WITH A FIXED POINT

(POLINOMIAL'NYE RESHENIIA URAVNENII DVIZHENIIA TELA, IMEIUSHCHEGO NEPODVIZHNUIU TOCHKU)

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In the present paper there are three exact solutions of the problem of motion of a rigid body with a fixed point which contains cavities filled with an ideal liquid, the five unknown variables are algebraic functions of the sixth one, whose dependence on time is determined by quadratures. The first two solutions are generalizations of the integrability case of Steklov [1] and Kowalewski [2], and the third solution coincides with the solution which was previously obtained by Goriachev in [3]. In the last case the motion of the body has the kinematic explanation based on the equation of a stationary hodograph [4].

1. Equations of motion of a body. We shall use a special rectangular coordinate system [5] fixed in a body. The first axis passes through the fixed point and through the center of gravity. The second and third axes are so selected that in the expression for the kinetic energy, which is the quadratic form component in x, y, z of the angular momentum

$$2T = ax^2 + a_1y^2 + a_2z^2 + 2(b_1y + b_2z)x$$

the term containing the product yz is absent. In such a coordinate system the problem is reduced to two first order equations [6]

$$\begin{gathered}
[(a_2z + b_2x)(y + \lambda_1) - (a_1y + b_1x)(z + \lambda_2)]\,[(y + \lambda_1)\,dz/dx - \\
-(z + \lambda_2)\,dy/dx] + (ax + b_1y + b_2z)\,[(y + \lambda_1)^2 + (z + \lambda_2)^2] + \\
+(x + \lambda)\,[{}^1/_2(ax^2 + a_1y^2 + a_2z^2) + (b_1y + b_2z)x - (a_1y + b_1x)\times \\
\times(y + \lambda_1) - (a_2z + b_2x)(z + \lambda_2) - E] - k = 0 \qquad (1.1)
\end{gathered}$$

$$\begin{gathered}
\{[(a_2z + b_2x)(y + \lambda_1) - (a_1y + b_1x)(z + \lambda_2)]\,dy/dx - \\
-(ax + b_1y + b_2z)(z + \lambda_2) + (a_2z + b_2x)(x + \lambda)\}^2 + \\
+\{[(a_1y + b_1x)(z + \lambda_2) - (a_2z + b_2x)(y + \lambda_1)]\,dz/dx - \\
-(ax + b_1y + b_2z)(y + \lambda_1) + (a_1y + b_1x)(x + \lambda)\}^2 + \\
+[{}^1/_2(ax^2 + a_1y^2 + a_2z^2) + (b_1y + b_2z)x - E]^2 - \Gamma^2 = 0
\end{gathered}$$

Here k and E are, respectively, the area constant and energy constant, Γ is the weight of the gyrostat multiplied by the distance between the center of gravity and the fixed point, the constants λ, λ_1, λ_2 are the components of the angular momentum of the liquid in a circulatory motion.

If from Equations (1.1)

$$y = y(x), \qquad z = z(x)$$

are found, then the dependence of the remaining variables on x are given by Formulas

$$\begin{aligned} \gamma &= {}^1/_2 (ax^2 + a_1y^2 + a_2z^2) + (b_1y + b_2z)\, x - E \\ \gamma' &= (ax + b_1y + b_2z)(y + \lambda_1) - (a_1y + b_1x)(x + \lambda) + \\ &\quad + [(a_2z + b_2x)(y + \lambda_1) - (a_1y + b_1x)(z + \lambda_2)]\, dz / dx \qquad (1.2) \\ \gamma'' &= (ax + b_1y + b_2z)(z + \lambda_2) - (a_2z + b_2x)(x + \lambda) + \\ &\quad + [(a_1y + b_1x)(z + \lambda_2) - (a_2z + b_2x)(y + \lambda_1)]\, dy / dx \end{aligned}$$

Here

$$\gamma = \Gamma\nu_1, \qquad \gamma' = \Gamma\nu_2, \qquad \gamma'' = \Gamma\nu_3 \qquad (1.3)$$

and ν_1, ν_2, ν_3 are the components of the unit vector in the direction of the force of gravity. The dependence of x on t is established by the quadrature

$$\frac{dx}{dt} = (a_2z + b_2x)(y + \lambda_1) - (a_1y + b_1x)(z + \lambda_2) \qquad (1.4)$$

Equations (1.1) become considerably simplified if the principal axes coincide with the special axes ($b_1 = b_2 = 0$), and the vector λ is directed along the first coordinate axis ($\lambda_1 = \lambda_2 = 0$). Using the customary notation for the principal axes $x = Ap$, $y = Bq$, $z = Cr$, $a = A^{-1}$, $a_1 = B^{-1}$, $a_2 = C^{-1}$, we can rewrite (1.1), (1.2), (1.4) as follows:

$$\begin{aligned} &[2Q + (A - B)p^2 + 2\lambda p]\frac{dR}{dp} - [2R + (A - C)p^2 + 2\lambda p]\frac{dQ}{dp} + \\ &\quad + (Ap + \lambda)\left[Q - R + \frac{(C - B)E}{A}\right] = \frac{(B - C)k}{A} \qquad (1.5) \end{aligned}$$

$$\begin{aligned} &B(B - C)[2Q + (A - B)p^2 + 2\lambda p]\left(\frac{dR}{dp}\right)^2 + C(C - B)[2R + \\ &+ (A - C)p^2 + 2\lambda p]\left(\frac{dQ}{dp}\right)^2 + ABC\left[Q - R + \frac{(C - B)E}{A}\right]^2 = \Gamma^2\frac{(B - C)^2BC}{A} \end{aligned}$$

$$\gamma = \frac{(Q - R)A}{B - C} - E, \qquad \gamma' = q\frac{dQ}{dp}, \qquad \gamma'' = r\frac{dR}{dp} \qquad (1.6)$$

$$Adp / dt = (B - C)\, qr \qquad (1.7)$$

The variables Q and R are replacing q and r

$$\begin{aligned} \frac{r^2(B - C)C}{A} &= (A - B)p^2 + 2\lambda p + 2Q \\ q^2\frac{(C - B)B}{A} &= (A - C)p^2 + 2\lambda p + 2R \end{aligned} \qquad (1.8)$$

Differentiating (1.5) with respect to p we obtain

$$[2R+(A-C)p^2+2\lambda p]\frac{d^2Q}{dp^2}+\left[\frac{dR}{dp}+(A-C)p+\lambda\right]\frac{dQ}{dp}+$$
$$+Bp\frac{dR}{dp}-\frac{(Q-R)AB}{B-C}+BE=0 \qquad (1.9)$$
$$[2Q+(A-B)p^2+2\lambda p]\frac{d^2R}{dp^2}+\left[\frac{dQ}{dp}+(A-B)p+\lambda\right]\frac{dR}{dp}+$$
$$+Cp\frac{dQ}{dp}-\frac{(Q-R)AC}{B-C}+CE=0 \qquad (1.10)$$

2. The conditions for the existence of polynomial solutions. We shall introduce the polynomials

$$Q=b_0+b_1p+b_2p^2+\ldots+b_np^n \qquad (b_n\neq 0) \qquad (2.1)$$
$$R=c_0+c_1p+c_2p^2+\ldots+c_mp^m \qquad (n\geqslant m) \qquad (2.2)$$

which satisfy Equations (1.5).

If we let $m>2$, and if in the identity obtained from substituting (2.1), (2.2) into (1.10) we set the coefficient of p^{2n+m-2} equal to zero, then we obtain

$$2n^2C\,(C-B)\,c_mb_n^2=0 \qquad (2.3)$$

Equation $C=B$ is not used in the derivation of (1.1), consequently from (2.1) and (2.3) we have that $c_m=0$ for all $m>2$, that is

$$R=c_0+c_1p+c_2p^2 \qquad (2.4)$$

The polynomial (2.1) cannot be of higher degree than four

$$Q=b_0+b_1p+b_2p^2+b_3p^3+b_4p^4 \qquad (2.5)$$

The polynomials (2.4) and (2.5) should turn (1.9), (1.10) into identities, which leads to the following conditions

$$b_4\,[2c_2+A-C-AB\,/\,16\,(B-C)]=0$$
$$b_4\,[12c_2+4C-AC\,/\,(B-C)]=0$$
$$28b_4\,(c_1+\lambda)+9b_3\,[2c_2+A-C-AB\,/\,9\,(B-C)]=0$$
$$4b_4c_1+b_3\,[10c_2+3C-AC\,/\,(B-C)]=0$$
$$24b_4c_0+15b_3\,(c_1+\lambda)+4b_2\,(2c_2+A-C)+2c_2B-$$
$$-(b_2-c_2)\,AB\,/\,(B-C)=0 \qquad (2.6)$$
$$3b_3c_1+4c_2\,(2b_2+A-B)+2b_2C-(b_2-c_2)\,AC\,/\,(B-C)=0$$
$$12b_3c_0+6b_2\,(c_1+\lambda)+b_1\,(2c_2+A-C)+c_1B-(b_1-c_1)AB\,/\,(B-C)=0$$
$$6c_2\,(b_1+\lambda)+c_1\,(2b_2+A-B)+b_1C-(b_1-c_1)\,AC\,/\,(B-C)=0$$
$$4b_2c_0+b_1\,(c_1+\lambda)=BH, \qquad 4c_2b_0+c_1\,(b_1+\lambda)=CH$$

The constant H replaces E

$$H=(b_0-c_0)\,A\,/\,(B-C)-E \qquad (2.7)$$

Substituting (2.4) and (2.5) into (1.8) and (1.6) and taking into account (2.7), we have

$$q^2B(C-B)/A = (2c_2 + A - C)p^2 + 2(c_1+\lambda)p + 2c_0 \qquad (2.8)$$
$$r^2C(B-C)/A = 2b_4p^4 + 2b_3p^3 + (2b_2 + A - B)p^2 + 2(b_1+\lambda)p + 2b_0$$
$$\gamma = [A/(B-C)]\,[b_4p^4 + b_3p^3 + (b_2-c_2)p^2 + (b_1-c_1)p] + H$$
$$\gamma' = q(4b_4p^3 + 3b_3p^2 + 2b_2p + b_1), \qquad \gamma'' = r(2c_2p + c_1)$$

The dependence of p on t is found from (1.7)

$$BC(dp/dt)^2 = [(2c_2 + A - C)p^2 + 2(c_1+\lambda)p + 2c_0]\,[2b_4p^4 + 2b_3p^3 + + (2p_2 + A - B)p^2 + 2(b_1+\lambda)p + 2b_0] \qquad (2.9)$$

Since Formulas in (1.5) are the first integrals of Equations (1.9) and (1.10), then the relations following from the requirement that (2.4) and (2.5) make (1.5) an identity, turn out to be the consequence of conditions (2.6), with the exception of the relations constraining the constants k and Γ :

$$k = \lambda H + \frac{2(b_0c_1 - c_0b_1)A}{B-C}, \qquad \Gamma^2 = H^2 + 2\left(\frac{b_0c_1^2}{C} - \frac{c_0b_1^2}{B}\right)\frac{A}{B-C} \qquad (2.10)$$

The three different solutions of Equations (2.6) lead to the three particular cases of integrability of the equations of motion of a body with a fixed point. These solutions are given in the Sections 3 to 5.

3. The first solution ($n = 2$). When $b_4 = b_3 = 0$ then Equations (2.6) are satisfied by

$$b_2 = \frac{(A-B)(A-C)}{2(2C-A)}, \qquad c_2 = \frac{(A-B)(A-C)}{2(2B-A)}$$

$$b_1 = \lambda\left[C\frac{3BC - AC - B^2}{(2C-A)^2(2B-A)} - 1\right], \qquad c_1 = \lambda\left[B\frac{3BC - AB - C^2}{(2B-A)^2(2C-A)} - 1\right]$$

$$b_0 = \frac{C(2B-A)}{2(A-B)(A-C)}\left\{H - \lambda^2\frac{3BC - AC - B^2}{(2C-A)^3(2B-A)^3}[A^3 - 2A^2(2B+C) + + AB(3B+8C) - BC(5B+C)]\right\} \qquad (3.1)$$

$$c_0 = \frac{B(2C-A)}{2(A-B)(A-C)}\left\{H - \lambda^2\frac{3BC - AB - C^2}{(2B-A)^3(2C-A)^3}[A^3 - 2A^2(2C+B) + + AC(3C+8B) - BC(5C+B)]\right\}$$

Substituting (3.1) into (2.8) and (2.9) we obtain the first of the two solutions presented without derivation in [7]. The constants k and Γ are determined from (2.10) and (3.1) in terms of A, B, C, λ, H. It is natural, however, to assume that Γ is prescribed and to express H in terms of Γ. We have

$$H = \pm\left(\Gamma^2 + \lambda^4\frac{A(B-C)^2(B+C-2A)\,[2A^2 - 3A(B+C) + 4BC]^3}{4(A-B)^2(A-C)^2(2B-A)^4(2C-A)^4}\right)^{1/2} + \lambda^2A\{[N/2(A-B)(A-C)(2B-A)^2(2C-A)^2] - 1\}/(2B-A)(2C-A)$$

where

$$N = 2A^4BC + A^3(B+C)(B^2 - 8BC + C^2) + 26A^2B^2C^2 - - 12AB^2C^2(B+C) + 2B^2C^2(B+C)^2$$

The two values of the parameter H lead to the two variants of the derived solution. This has been noticed by Kuz'min [8] for the Steklov's solution [1] (which in this Section can be obtained by setting $\lambda = 0$).

4. The second solution ($n = 3$). Let $b_4 = 0$, $b_3 = b \neq 0$. If the quantities A, B, C are related by

$$A = 18C(B - C)/(10B - 9C)$$

then in this case the coefficients of the polynomials in (2.4), (2.5) are determined from (2.6).

We also find

$$c_2 = \frac{3}{2} C \frac{3C - 2B}{10B - 9C}, \qquad c_1 = -2\lambda - \frac{3C(3C - 2B)^2(3C - 4B)}{b(10B - 9C)^2}$$

$$c_0 = \lambda^2 \frac{10B - 9C}{4BC} - \frac{3}{2}\frac{\lambda}{b}\frac{(3C - 2B)(2B - C)}{10B - 9C} - \frac{81C(B - C)^2(3C - 2B)^3(3C - 4B)}{b^2 B(10B - 9C)^3}$$

$$b_2 = -\frac{3(3C - 2B)(27C^2 - 54BC + 22B^2)}{2B(10B - 9C)} - \frac{3}{2} b\lambda \frac{10B - 9C}{BC}$$

$$b_1 = \frac{3}{4} b\lambda^2 \frac{(10B - 9C)^2}{B^2C^2} + \frac{\lambda}{8B^2C}(729C^3 - 1917BC^2 + 1512B^2C - 388B^3) +$$
$$+ \frac{3}{8}\frac{(3C - 2B)^2(3C - 4B)}{bB^2(10B - 9C)}(243C^3 - 648BC^2 + 495B^2C - 122B^3)$$

$$b_0 = -\frac{b\lambda^3}{8}\frac{(10B - 9C)^3}{B^3C^3} - \frac{\lambda^2}{48B^3C^2}\frac{10B - 9C}{3C - 2B}(2187C^4 - 7533BC^3 + 9234B^2C^2 -$$
$$- 5036B^3C + 1064B^4) + \frac{\lambda}{4b}\frac{3C - 2B}{B^2C(10B - 9C)}(729C^4 - 2754BC^3 + 3951B^2C^2 -$$
$$- 2582B^3C + 632B^4) - \frac{3(B - C)(3C - 2B)^3(3C - 4B)}{16B^3(10B - 9C)^3}(2187C^4 - 5832BC^3 +$$
$$+ 4131B^2C^2 - 30B^3C - 488B^4)$$

Substituting these values into (2.8) and (2.9), we obtain the second solution, which is derived in [7].

5. The third solution ($n = 4$). Assuming that $b_4 \neq 0$, we require that the parameters of the system satisfy the following conditions

$$\lambda = 0, \quad A = 16C(B - C)/(9B - 8C) \tag{5.1}$$

Equations (2.6) give

$$c_2 = C\frac{4C - 3B}{9B - 8C}, \quad c_1 = 0, \quad c_0 = -\frac{HB^2(9B - 8C)}{8(4C - 3B)(2C - 3B)(2C - B)}$$

$$b_4 = \frac{2C(4C - 3B)^2(4C - 5B)(2C - 3B)(2C - B)}{HB^2(9B - 8C)^3}, \quad b_3 = 0 \tag{5.2}$$

$$b_2 = -\frac{2(4C - 3B)(2C - 3B)(2C - B)}{B(9B - 8C)}, \quad b_1 = 0, \quad b_0 = \frac{9B - 8C}{4(4C - 3B)}H$$

and from (2.8) we find

$$q^2 + \frac{16C^2}{(9B - 8C)^2}p^2 = \frac{4BCH}{(4C - 3B)(2C - 3B)(2C - B)} \tag{5.3}$$

$$r^2 = \frac{64C(4C - 3B)^3(4C - 5B)(2C - 3B)(2C - B)}{(9B - 8C)^2B^3H}p^4 - \tag{5.4}$$
$$- \frac{16(4C - 3B)(16C^2 - 28BC + 9B^2)}{(9B - 8C)^2B}p^2 + \frac{8H}{4C - 3B}$$

$$\gamma = \frac{32C^2(4C-3B)^3(4C-5B)(2C-3B)(2C-B)}{HB^2(9B-8C)^4}p^4 - \frac{16C(4C-3B)(8C^2-15BC+6B^2)}{B(9B-8C)^2}p^2 + H \tag{5.5}$$

$$\gamma' = \frac{8C(4C-3B)^3(4C-5B)(2C-3B)(2C-B)}{HB^2(9B-8C^3)}pq\left[p^2 - \frac{B(9B-8C)^2H}{2C(4C-3B)^2(4C-5B)}\right]$$

$$\gamma'' = 2C\frac{4C-3B}{9B-8C}pr$$

Goriachev [3] obtained the same solution in a different way. The condition $\lambda = 0$ is satisfied always when liquid filled cavities are absent. Besides $A < B+C$, $B < A+C$. This, together with (5.1) gives

$$0.375\,B < C < B \tag{5.6}$$

From (2.10) and (5.2) we have that $k = 0$, $H = -\Gamma$. The minus sign in the last relation was chosen for the following reason: when $H = \Gamma > 0$ then by (5.3) we have that $0.5B < C < 0.75B$, and all terms on the right-hand side of (5.4) are negative. The requirement that at $H = -\Gamma$ the quantities p, q, r satisfying (5.3) and (5.4) must be real, constrains the interval (5.6) into

$$0.375\,B < C < 0.5\,B \tag{5.7}$$

From (5.1) and (5.7) we conclude that $B > A > C$, which means that the center of gravity of the body is on the mean axis of the ellipsoid of inertia.

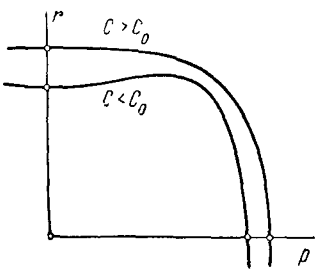

Fig. 1

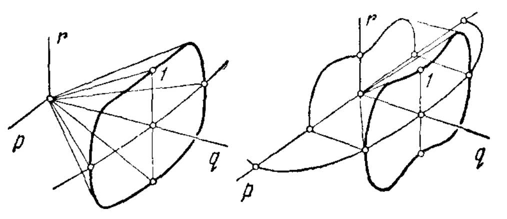

Fig. 2

Let us express the components of the angular velocity in $\sqrt{\Gamma/C}$ units and introduce the dimensionless parameter $c = C/B$. From (5.7) we have

$$0.375 < c < 0.5 \tag{5.8}$$

Introducing the new variable σ and remembering the notation in (1.3) we can write the solution (5.3) to (5.5) in the dimensionless form

$$p = \frac{9-8c}{2(3-4c)}\sqrt{\sigma}, \quad q = \frac{2c}{3-4c}\sqrt{\sigma^\circ - \sigma}, \quad r = 2\left(\frac{5-4c}{\sigma^\circ}(\sigma_0-\sigma)(\sigma_1+\sigma)\right)^{1/2} \tag{5.9}$$

$$\nu_1 = -1 + 4\frac{6-15c+8c^2}{3-4c}\sigma - 2\frac{5-4c}{\sigma^\circ}\sigma^2$$

$$\nu_2 = 2\frac{5-4c}{\sigma^\circ}\left(\frac{2}{5-4c}-\sigma\right)\sqrt{\sigma(\sigma^\circ-\sigma)}, \quad \nu_3 = -2\left(\frac{5-4c}{\sigma^\circ}\sigma(\sigma_0-\sigma)(\sigma_1+\sigma)\right)^{1/2} \tag{5.10}$$

Here

$$\sigma^\circ = \frac{3-4c}{(3-2c)(1-2c)}, \quad \sigma_0 = \frac{R(c) + 9 - 28c - 16c^2}{2(5-4c)(3-2c)(1-2c)}$$

$$\sigma_1 = \frac{R(c) - 9 + 28c - 16c^2}{2(5-4c)(3-2c)(1-2c)}, \quad R(c) = \sqrt{(3-4c)(27 - 92c + 96c^2 - 32c^3)}$$

In the interval (5.8)

$$0 < \sigma_0 < \sigma^\circ, \qquad 0 < \sigma_1 \tag{5.11}$$

The time dependence of σ can be obtained by substituting (5.9) in Equation (1.7) which has been previously put into the dimensionless form

$$d\sigma / d\tau = \sqrt{\sigma(\sigma_0 - \sigma)(\sigma^\circ - \sigma)(\sigma_1 + \sigma)} \qquad (\tau = t\sqrt{(5-4c)\Gamma / B\sigma^\circ c}) \tag{5.12}$$

From (5.12) we are able to conclude that σ is an elliptic function of time. Varying within its bounds

$$0 \leqslant \sigma \leqslant \sigma_0 \tag{5.13}$$

it moves from one bound to another in a finite time.

6. Kinematic explanation. The motion of a body with a fixed point can be regarded as the rolling without slip of a curve fixed in the body (the moving hodograph of the angular velocity) on another curve fixed in space (the stationary hodograph of the angular velocity).

In the case which was considered in the previous Section the moving hodograph is given by Equations (5.9). This curve is the intersection of the elliptic cylinder

$$q^2 + \frac{16c^2}{(9-8c)^2}p^2 = \frac{4c^2}{(3-4c)(3-2c)(1-2c)}$$

with the fourth order cylinder

$$r^2 = \frac{8c}{3-4c} + 16\frac{(3-4c)(9-28c+16c^2)}{(9-8c)^2}p^2 - 64\frac{(3-4c)^3(5-4c)(3-2c)(1-2c)}{(9-8c)^4}p^4 \tag{6.1}$$

The directrix of the cylinder (6.1) is symmetric with respect to p and r. When $c > c_0 = \frac{1}{8}(7 - \sqrt{13})$ this curve is concave, when $c < c_0$ it has inflection points (Fig.1). The corresponding moving hodographs are shown in Fig.2.

Let us introduce a stationary cylindrical coordinate system ζ, ρ, α, the ζ-axis along the unit vector ν (see (1.13)). The fixed hodograph of the angular velocity is given [4] by Equations

$$\omega_\zeta = p\nu_1 + q\nu_2 + r\nu_3, \qquad \omega_\rho^2 = p^2 + q^2 + r^2 - \omega_\zeta^2$$

$$\omega_\rho^2 \, d\alpha = (r\nu_2 - q\nu_3)\,dp + (p\nu_3 - r\nu_1)\,dq + (q\nu_1 - p\nu_2)\,dr$$

We shall substitute here (5.9) and (5.10)

$$\omega_\zeta = \frac{\sqrt{\sigma}}{3-4c}\left[-\frac{9-8c}{2} + 2c(5-4c)\sigma + (5-4c)(3-2c)(1-2c)\sigma^2\right] \tag{6.2}$$

$$\omega_\rho^2 = k_0 + k_1\sigma - k_2\sigma^2 + k_3\sigma^3 - k_4\sigma^4 - k_5\sigma^5 \tag{6.3}$$

$$\frac{d\alpha}{d\sigma} = \frac{\Pi(\sigma)}{n\omega_\rho^2(\sigma)\sqrt{(\sigma_0-\sigma)(\sigma^\circ-\sigma)(\sigma_1+\sigma)}} \tag{6.4}$$

Here

$$k_0 = \frac{4c\,(6 - 15c + 8c^2)}{(3 - 4c)\,(3 - 2c)\,(1 - 2c)}, \qquad k_1 = 4\,\frac{27 - 120c + 159c^2 - 64c^3}{(3 - 4c)^3}$$

$$k_2 = 2\,\frac{(5 - 4c)\,(18 - 81c + 96c^2 - 32c^3)}{(3 - 4c)^2}, \qquad k_3 = \frac{(5 - 4c)\,(27 - 96c + 80c^2 - 16c^3)}{(3 - 4c)^2}$$

$$k_4 = 4c\,\frac{(5 - 4c)^2\,(3 - 2c)\,(1 - 2c)}{(3 - 4c)^2}, \qquad k_5 = \frac{(5 - 4c)^2\,(3 - 2c)^2\,(1 - 2c)^2}{(3 - 4c)^2}$$

$$n = (3 - 4c)\,\sqrt{(3 - 4c)\,(5 - 4c)\,(3 - 2c)\,(1 - 2c)}$$

$$\Pi\,(\sigma) = n_0 - n_1\sigma - n_2\sigma^2 - n_3\sigma^3 + n_4\sigma^4 \qquad (6.5)$$

$$n_0 = 2c\,\frac{27 - 69c + 48c^2 - 8c^3}{(3 - 2c)\,(1 - 2c)}, \qquad n_1 = 3c\,\frac{(5 - 4c)\,(8c - 3)\,(7 - 16c + 8c^2)}{(3 - 2c)\,(1 - 2c)}$$

$$n_2 = 3\,(5 - 4c)\,(-18 + 107c - 176c^2 + 80c^3)$$

$$n_3 = (5 - 4c)\,(3 - 2c)\,(1 - 2c)\,(63 - 172c + 96c^2)$$

$$n_4 = 3\,(5 - 4c)^2\,(3 - 2c)^2\,(1 - 2c)^2$$

The fixed hodograph (6.2) to (6.4) is on the surface of revolution whose meridian is given by Equations (6.2) and (6.3).

Constructing this meridian we take into account that in the interval (5.13)

$$\omega_\zeta\,(0) = 0, \quad \omega_\zeta\,(\sigma_*) = 0, \quad \min \omega_\zeta\,(\sigma) = \omega_\zeta\,(\sigma^*) < 0, \qquad 0 < \sigma^* < \sigma_* < \sigma_0$$

besides

$$\sigma_* = \{\sqrt{27 - 96c + 110c^2 - 40c^3} - c\,\sqrt{2\,(5 - 4c)}\}\,\{(3 - 2c)\,(1 - 2c)\,\sqrt{2\,(5 - 4c)}\}^{-1}$$

$$\sigma^* = \{\sqrt{135 - 480c + 590\,c^2 - 232c^3} - 3c\,\sqrt{2\,(5 - 4c)}\}\,\{(5\,(3 - 2c)\,(1 - 2c)\,\sqrt{2\,(5 - 4c)}\}^{-1}$$

The magnitude of the angular velocity is at its minimum in the left-hand side of the interval (5.13) and at its maximum when

$$\sigma = \sigma_{**} = (171 - 460c + 256c^2)\,\{32\,(5 - 4c)\,(3 - 2c)\,(1 - 2c)\}^{-1}$$

$\sigma_{**} < \sigma_*$ when c is close to 0.375 (6.6)

$\sigma_{**} > \sigma_0$ when c is close to 0.5 (6.7)

Fig.3 shows the part of the meridian which corresonds to the positive sign of the radical in (6.2). By changing the sign of the radical we obtain

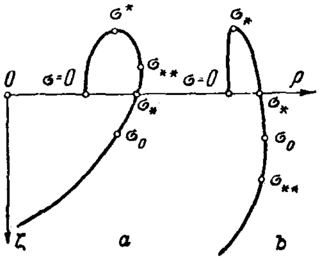

Fig. 3

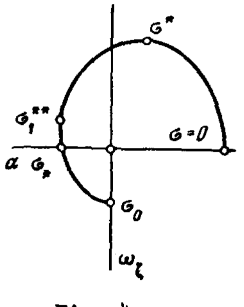

Fig. 4

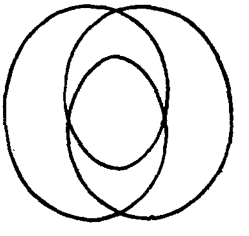

Fig. 5

the remaining part of the meridian, a curve symmerical with respect to the ρ-axis, and also shown in Fig.3. Fig.3*a* corresponds to the case (6.6), Fig. 3*b* to the case (6.7).

In the interval (5.8) the quantities n_i are positive, consequently the number of changes of sign of the coefficients of the polynomial (6.5) equals two. Thus the number of positive roots of this polynomial is not greater than two, and since $\Pi(0) > 0$ and

$$\Pi(\sigma_0) = -(5-4c)(3-2c)(1-2c)(\sigma^\circ-\sigma_0)(\sigma_1+\sigma_0)\times$$
$$\times\{2c+(5-4c)(3-2c)(1-2c)\sigma_0^2\} < 0$$

we have only one root in the interval (5.13). Let us denote it by σ^{**}

$$\Pi(\sigma^{**}) = 0, \qquad 0 < \sigma^{**} < \sigma_0$$

From (5.11) and (5.13) it follows that the radical $\sqrt{(\sigma^\circ-\sigma)(\sigma_1+\sigma)}$ does not change the sign in the interval (5.13). Let us assume that this radical is positive, and that the initial instant of time is $\sigma = 0$. From (5.12) and (5.13) we can find that at instants of time which are close to the initial instant $d\sigma/d\tau > 0$. Consequently, the sign of the radical $\sqrt{\sigma}$ and of $\sqrt{\sigma_0-\sigma}$ are both the same and both are positive.

Let us consider the curve (6.2), (6.4). When σ is increasing, the variable ω_ζ is decreasing from zero to its minimum value $\omega_\zeta(\sigma^*)$, and then it is increasing to the value $\omega_\zeta(\sigma_0)$, and vanishing at $\sigma = \sigma_*^\circ$. The angle α increases, reaching its maximum at $\sigma = \sigma^{**}$, then decreases to the value $\alpha(\sigma_0)$. The corresponding part of the curve investigated is shown in Fig.4.

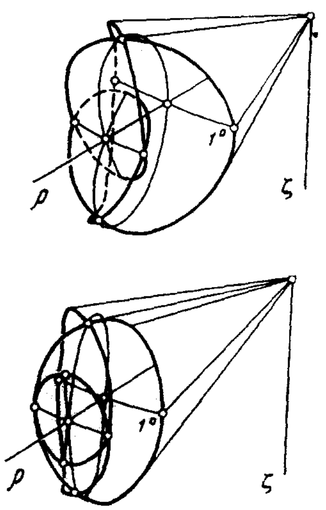

Fig. 6

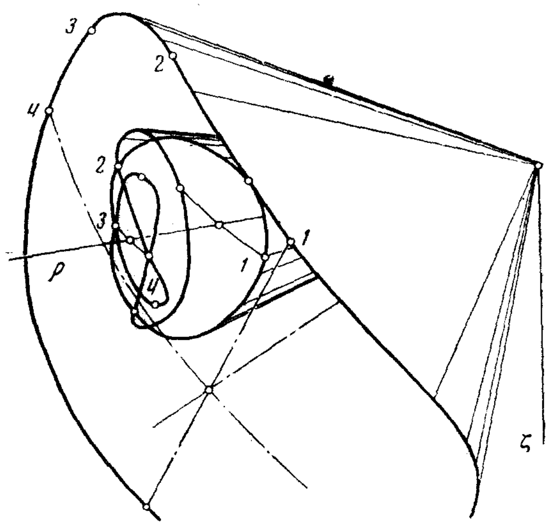

Fig. 7

When $\sigma \to \sigma_0$, the radical $\sqrt{\sigma_0-\sigma}$ changes sign and $d\sigma/d\tau$ becomes negative, σ decreases from σ_0 to zero. The part of the curve corresonding to this stage is symmetric to the curve shown in Fig.4 about the vertical line through the point determined by the value σ_0. After the successive variations of σ from zero to σ_0, and then from σ_0 to zero, we obtain the remaining part of the curve (Fig.5). When we transfer it on the surface of revolution whose meridian is shown in Fig.3, we obtain the fixed hodograph shown in Fig.6.

By (5.9) and (5.10) we find out that when $\sigma = 0$, then $p = 0$, $\nu_1 = -1$,

$\nu_2 = \nu_3 = 0$. Hence, at the initial instant the p-axis is vertical, the center of gravity is above the point of support, the tip of the angular velocity vector coincides with the point 1 of the moving hodograph (Fig.2). The corresponding point on the fixed hodograph is denoted by the same numeral 1 (Fig.6). In the successive motion of the body when the moving hodograph rolls on the fixed hodograph, the points of these curves which come into contact with each other are marked, respectively, by 2,3,4 and so on (Fig.7).

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